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ON THE DEFINITION OF AN INFINITE NUMBER.

By DR. G. A. MILLER.

Mathematics is replete with surprises. Three of these are especially noteworthy on account of the fundamental principles involved, namely, (1) There are perfectly consistent geometries in which the sum of the angles of a plane triangle is not equal to two right angles. (2) There are algebras in which the commutative law of multiplication does not hold. (3) There are aggregates such that a part is equivalent to the whole.

The first of these has been frequently noted in this journal. In regard to the second, Poincaré recently said*: "Hamilton's quaternions give us an example of an operation which presents an almost perfect analogy with multiplication, which may be called multiplication, and yet it is not commutative; *i. e.*, the product is changed when the order of the factors is changed. This presents a revolution in arithmetic which is entirely similar to the one which Lobatchevski effected in Geometry."

The object of the present note is to give some details in regard to an infinite aggregate. The method employed to compare two aggregates is one of the most rudimentary ones. In fact, it is essentially the same as that employed in the earliest stages of civilization when objects are counted by means of pebbles. In this way a (1, 1) correspondence is established between a given pile of pebbles and a certain aggregate of objects.

Similarly, two aggregates or totalities are said to be *equivalent* when it is possible to establish a (1, 1) correspondence between the units of the aggregates. That is, when the units of the aggregates can be so associated that to each unit of one of the aggregates there corresponds one and only one unit of the other and *vice versa.* For instance, every even integer is of the form $2n$, and every odd integer is of the form $2n+1$. For any integral value of n we obtain one and only one even number, and also one and only one odd number. By associating these two numbers for every value of n we obtain a (1, 1) correspondence between the odd and the even numbers. Hence we say that the totality of even numbers is equivalent to the totality of odd numbers.

A more interesting illustration may be given by means of the equation $y=1/(x+1)$. Suppose we assign to x any positive rational value. The corresponding value of y will always be a positive rational number which does not exceed unity. Whenever we give x two distinct values, y will assume two distinct values in the given intervals and *vice versa.* Hence it follows that *the rational positive numbers which do not exceed unity are equivalent to all the rational positive numbers.*

By employing the equation $2y=1/(x+1)$, it is easy to see that the totality of the positive rational numbers which do not exceed one-half is equivalent to

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*Poincare, *Bulletin des Sciences Mathematiques*, Vol. 26, 1902, page 250.

the totality of all the positive rational numbers. More generally, the equation $ay=1/(x+1)$, a being a positive rational number, indicates that the totality of the positive rational numbers which do not exceed an arbitrary finite limit is equivalent to the totality of all the positive rational numbers.

The above examples furnish some of the most important instances of multitudes in which a part is equivalent to the whole. It should be observed that this equivalence is proved by one of the most elementary methods of comparison—a method which every one employs with perfect confidence.

With respect to any aggregate only two cases are thinkable; either a part is equivalent to the whole or there is no such part. In the former case the aggregate is said to be infinite, in the latter case it is said to be finite. This definition of an infinite number of things is due to Dedekind and is of fundamental importance.* That infinite numbers have this property was noted by others, but Dedekind was the first to use it as a definition.

In employing this definition any aggregate (M_1) is to be considered a part of another aggregate (M_2), provided each of the units of M_1 is contained in M_2, while there is at least one unit in M_2 which is not contained in M_1. For instance, all the even integers are a part of the total number of integers and all the integers are a part of all the rational numbers.

It is now easy to prove that the number of integers is infinite, for the equation $y=2x$ shows that there is just one even integer (y) which corresponds to any integer (x), and that there is just one integer x which corresponds to any even number y. That is, there is a (1, 1) correspondence between the total number of integers and the even integers. As the latter constitute a part of the former it follows that a part of the totality of integers is equivalent to the whole number of integers, that is, the number of integers is infinite.†

By the same method it may be proved that time, according to the common conception, is infinite; for there is a (1, 1) correspondence between the total number of hours and the total number of half hours. If the half hours were denoted by the natural numbers the hours would correspond to the even numbers. Similarly, it may be observed that space (defined analytically by coördinates) is infinite according to the given definition.

If it is possible to establish a (1, 1) correspondence between two infinite totalities they are said to be of the same power. The totality (M_0) formed by the natural numbers is especially important. Any totality which has the same power as M_0 is said to be numerable (countable). It is an extremely interesting fact that all algebraic numbers are countable. In particular, the natural numbers are equivalent to all the rational numbers. These facts constitute some of the elements of the important subject known as the theory of aggregates (ensemble, Mengenlehre) to which the reader may be referred for further developments.

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*Dedekind, *Essays on Number*, translated by Beman, 1901, page 63.

†This result can be proved in an infinite number of ways by using the equation $y=ax$, a being any integer. It may also be proved by the equations $y=x^a$, $y=x+a$, etc.